# Lecture 8.

# Independent Events.

- Suppose you play the lottery regularly and you have a set of lucky numbers that you play every time.

- Suppose you play 100 times and lose every time. Do playing your "lucky numbers" give you an edge on the $101^{st}$ play?

- It is not uncommon for people to think "all numbers will come up eventually" and since mine haven't come up yet, I have a better chance next time".

- While this reasoning would be valid if past numbers were no longer valid, it fails with replacement. This is an example of independence.

- A simpler example: you toss a coin twice. Say $H_i$ = the ith toss is heads.
- What is $P(H_2 | H_1) = ?$

$P(H_2|H_1) = 1/2$, since the first toss has no affect on the outcome of the second toss.

- In particular, $P(H_2|H_1) = P(H_2) = 1/2$.
- Furthermore, $P(H_1 H_2) = P(H_1)P(H_2) = 1/4$.

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**Defn** We say that $E$ is *independent* of $F$ if $P(E|F) = P(E)$.

**Prop:** TFAE:

1) $P(E|F) = P(E)$.
2) $P(F|E) = P(F)$.
3) $P(EF) = P(E)P(F)$
4) $P(E^c F^c) = P(E^c)P(F^c)$

**Pf:** $1 \Rightarrow 2), 3)$

$$P(E) = P(E|F) = \frac{P(EF)}{P(F)}$$

So $P(E)P(F) = P(EF)$ (this gives 3))

Hence $P(F) = \dfrac{P(FE)}{P(E)} = P(F|E)$ (2). ✓

3) $\Rightarrow$ 4)

$$
\begin{aligned}
P(E^cF^c) &= P((E \cup F)^c) \\
&= 1 - P(E \cup F) \\
&= 1 - P(E) - P(F) + P(EF) \\
&= 1 - P(E) - P(F) + P(E)P(F) \text{ (by 3))} \\
&= (1 - P(E))(1 - P(F)) \\
&= P(E^c)P(F^c).
\end{aligned}
$$

4 $\Rightarrow$ 3 by symmetry, 2 $\Rightarrow$ 1 by symmetry.

So it suffices to show 3 $\Rightarrow$ 2.

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\begin{aligned}
P(F|E) &= \frac{P(EF)}{P(E)} \\
&= \frac{\cancel{P(E)}P(F)}{\cancel{P(E)}} \quad \text{by 3.} \\
&= P(F) \qquad \square.
\end{aligned}
$$

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Prop: If $E$ and $F$ are indep, then so are $E$ and $F^c$.

Pf: $E = EF \cup EF^c$ and $EF \cap EF^c = \emptyset$,

So

$$P(E) = P(EF) + P(EF^c)$$

$$P(E) = P(E)P(F) + P(EF^c) \quad \text{(since } E \text{ and } F \text{ are indep).}$$

So $P(E) - P(E)P(F) = P(EF^c)$

$$P(E)(1 - P(F)) = P(EF^c)$$

$$P(E)P(F^c) = P(EF^c)$$

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More generally: We say events $E_1, \dots, E_n$ are independent iff for all $1 \le r \le n$, and all $1 \le i_1 < \dots < i_r \le n$

$$P(E_{i_1} \cdots E_{i_r}) = P(E_{i_1}) \cdots P(E_{i_r}).$$

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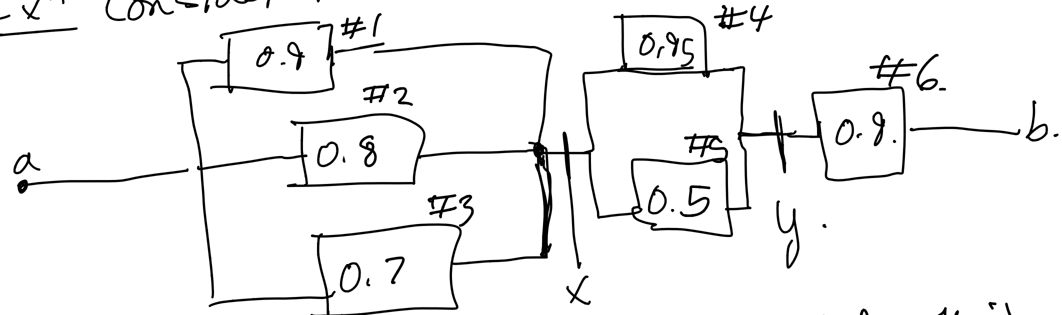
Remark on independence and Mutual Exclusivity

Suppose $E$ and $F$ are events such that $P(E) \neq 0$ and $P(F) \neq 0$, which are mutually exclusive. Then they <u>cannot</u> be independent: For a contradiction, suppose otherwise. Then

$$0 = P(\emptyset) = P(EF) = P(E)P(F) \neq 0.$$

Ex: Consider the circuit




- Assume that each device is labelled with its probability of being functional, and the devices fail independently.

What is the probability of a path from a to b?
Divide into three circuits. There is a path from a to b iff there is a path from a to x and a path from x to y and a path from y to b.

Let $T_i$ = "Device $\#i$ is functional."

So there is only a path from A to x iff $T_1 \cup T_2 \cup T_3$.

$$P(T_1 \cup T_2 \cup T_3) = 1 - P(T_1^c T_2^c T_3^c)$$

$$= 1 - P(T_1^c)P(T_2^c)P(T_3^c)$$
$$= 1 - (1-0.9)(1-0.8)(1.0.7)$$
$$= 0.994$$

similarly:

$$P(T_4 \cup T_5) = 1 - P(T_4^c)P(T_5^c)$$
$$= 1 - (1-0.85)(1-0.5) = 0.525$$

$$P(T_6) = 0.9$$

Then, since all devices are indep.

$$P(a \text{ to } b) = P((T_1 \cup T_2 \cup T_3) \cap (T_4 \cup T_5) \cap T_6)$$
$$= P(T_1 \cup T_2 \cup T_3)\,P(T_4 \cup T_5)\,P(T_6)$$
$$= (0.994)(0.525)(0.9)$$
$$= 0.465 \equiv 46.5\%$$